EQUILIBRIUM EQUATIONS OF A PLATE OF VARIABLE THICKNESS

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The symbolic method of Lur'e [1] and the principle of minimum potential energy are used in our derivation of the differential equilibrium equations of a plate of variable thickness and of boundary conditions. Rectangular plates and axially symmetrical circular plates are considered. The equilibrium equations and boundary conditions were derived (in Cartesian coordinates) in [2, 3] for a plate of uniform thickness.

1. Derivation of the equilibrium equations of the plate in Cartesian coordinates. Let u_0, v_0, w_0 be the displacements of the points of an original plane $z = 0$, and u_0', v_0', w_0' be the values of the derivatives of displacements along coordinate z in this plane; we have then [1]

$$u = cu_0 - \frac{mzs\partial_1\vartheta_0}{2(m-2)} + su_0' - \frac{m\lambda\partial_1\vartheta_0'}{4(m-1)}$$
$$v = cv_0 - \frac{mzs\partial_2\vartheta_0}{2(m-2)} + sv_0' - \frac{m\lambda\partial_2\vartheta_0'}{4(m-1)} \qquad (1.1)$$
$$w = sw_0' + \frac{m\lambda\Delta\vartheta_0}{2(m-2)} + cw_0 - \frac{mzs\vartheta_0'}{4(m-1)}$$

Here [2]

$$c = \cos zD = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}\Delta^n}{(2n)!}, \qquad s = \frac{\sin zD}{D} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}\Delta^n}{(2n+1)!}$$

$$\lambda = \frac{s - zc}{\Delta} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+3}\Delta^n}{(2n+1)!\,(2n+3)}, \qquad \Delta = D^2 = \partial_1^2 + \partial_2^2 \qquad (1.2)$$

$$\partial_1 = \frac{\partial}{\partial x}, \quad \partial_2 = \frac{\partial}{\partial y}, \quad \vartheta_0 = \partial_1 u_0 + \partial_2 v_0 + w_0', \quad \vartheta_0' = \partial_1 u_0' + \partial_2 v_0' - \Delta w_0$$

The variation of the specific potential energy of the plate deformation is

$$\delta\pi = \sigma_x\delta e_x + \sigma_y\delta e_y + \sigma_z\delta e_z + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx} \qquad (1.3)$$

Let us express the variations of deformations in terms of variations of the basic variables u_0, v_0, w_0, u_0', v_0', w_0', making use of the relations between displacements and deformations and of equations (1.1). We have, for instance,

$$\delta e_x = c\partial_1\delta u_0 - \frac{mzs\partial_1^2}{2(m-2)}\delta\vartheta_0 + s\partial_1\delta u_0' - \frac{m\lambda\partial_1^2}{4(m-1)}\delta\vartheta_0' \qquad (1.4)$$

Expressions for other variations of deformations will not be given here; they can be found in [3] (formulas (1.2) and (2.1)).

To obtain the potential energy of the plate deformation, the specific potential energy must be integrated over the plate volume, i.e. over its thickness and the area of its base in plan projection. Let us first integrate over the thickness; let the top and bottom surfaces of the plate be given by equations $z = h_1(x, y)$ and $z = h_2(x, y)$. We obtain the following expressions:

$$\int_{h_2}^{h_1} \sigma_x \delta e_x dz = \sum_{n=0}^{\infty} (T_x^{(n)} \partial_1 \delta \chi_x^{(n)} + G_x^{(n)} \partial_1 \delta \psi_x^{(n)})$$

$$\int_{h_2}^{h_1} \tau_{xy} \delta \gamma_{xy} dz = \sum_{n=0}^{\infty} \{S^{(n)} \delta (\partial_1 \chi_y^{(n)} + \partial_2 \chi_x^{(n)}) + H^{(n)} \delta (\partial_1 \psi_y^{(n)} + \partial_2 \psi_x^{(n)})\} \qquad (1.5)$$

$$\int_{h_2}^{h_1} \sigma_z \delta e_z dz = \sum_{n=0}^{\infty} (Z_j^{(n)} \delta \xi^{(n)} + Z_i^{(n)} \delta \varphi^{(n)}) \quad \text{etc.}$$

We have introduced here static $(T_x^{(0)}, \ G_x^{(0)}, \ldots)$ and hyperstatic $(T_x^{(1)}, T_x^{(2)} \ldots, G_x^{(1)}, G_x^{(2)}, \ldots)$ stress characteristics, conforming to the formulas (1.3), (1.4), (2.2) and (2.3) in [3]; the values $\chi_x^{(n)}, \ \chi_y^{(n)}, \ \xi^{(n)}, \psi_x^{(n)} \ \psi_y^{(n)}, \ \varphi^{(n)}$ are also introduced (see formulas (1.6) and (2.5) of [3]). Let us note that

$$\chi_x^{(0)} = u_0, \quad \chi_y^{(0)} = v_0, \quad \xi^{(0)} = w_0, \quad \psi_x^{(0)} = u_0', \quad \psi_y^{(0)} = v_0', \quad \varphi^{(0)} = w_0' \qquad (1.6)$$

Let us add up all the integrals of type (1.5) and then integrate the obtained expression over the area of the plate base in plan projection Ω. Some of the double integrals over the area Ω will then change into integrals along contour L embracing area Ω; in this manner

$$\begin{aligned}\delta\Pi = \sum_{n=0}^{\infty} \Big\{ \oint_{(L)} [(n_x T_x^{(n)} + n_y S^{(n)}) \delta \chi_x^{(n)} + (n_y S^{(n)} + n_y T_y^{(n)}) \delta \chi_y^{(n)} + (n_x N_x^{(n)} + n_y N_y^{(n)}) \delta \xi^{(n)} + \\ + (n_x G_x^{(n)} + n_y H^{(n)}) \delta \psi_x^{(n)} + (n_x H^{(n)} + n_y G_y^{(n)}) \delta \psi_y^{(n)} + (n_x \Gamma_x^{(n)} + n_y \Gamma_y^{(n)}) \delta \varphi^{(n)}] \, ds - \\ - \iint_{(\Omega)} [(\partial_1 T_x^{(n)} + \partial_2 S^{(n)} + \Gamma_x^{(n-1)}) \delta \chi_x^{(n)} + (\partial_1 S^{(n)} + \partial_2 T_y^{(n)} + \Gamma_y^{(n-1)}) \delta \chi_y^{(n)} + \\ + (\partial_1 N_x^{(n)} + \partial_2 N_y^{(n)} - Z_j^{(n-1)}) \delta \xi^{(n)} + (\partial_1 G_x^{(n)} + \partial_2 H^{(n)} - N_x^{(n)}) \delta \psi_x^{(n)} + \\ + (\partial_1 H^{(n)} + \partial_2 G_y^{(n)} - N_y^{(n)}) \delta \psi_y^{(n)} + (\partial_1 \Gamma_x^{(n)} + \partial_2 \Gamma_y^{(n)} - Z_i^{(n)}) \delta \varphi^{(n)}] \, dx dy \Big\} \qquad (1.7)\end{aligned}$$

$$(\Gamma_x^{(-1)} = \Gamma_y^{(-1)} = Z_j^{(-1)} = 0)$$

Let us calculate the elementary work δA_1 of external forces applied to the faces of the plate. We shall denote the vectors of external forces applied to unit areas of the top and bottom surfaces of the plate as $\mathbf{P}_1$ and $\mathbf{P}_2$. We have

$$\delta A_1 = \iint_{(\Omega_1)} \mathbf{p}_1 \cdot \delta \mathbf{u}_1 d\Omega_1 + \iint_{(\Omega_2)} \mathbf{p}_2 \cdot \delta \mathbf{u}_2 d\Omega_2 \qquad (1.8)$$

Here Ω_1 and Ω_2 are the surface areas of the top and bottom bases of the plate, and $\mathbf{u}_1$, $\mathbf{u}_2$ are the displacement vectors of the points on these surfaces. Further,

$$d\Omega_1 = \frac{dx\,dy}{\cos(z, \mathbf{n}_1)}, \qquad d\Omega_2 = \frac{dx\,dy}{|\cos(z, \mathbf{n}_2)|} \qquad (1.9)$$

where $(z, \mathbf{n}_i)$ is the angle between the z-axis and the external normal to the surface $z = h_i(x,y)$ $(i = 1,2)$. Since

$$|\cos(z, \mathbf{n}_i)| = \frac{1}{\sqrt{1 + (\partial_1 h_i)^2 + (\partial_2 h_i)^2}} = \frac{1}{D_i(x, y)} \qquad (i = 1, 2) \qquad (1.10)$$

and allowing for relations (1.9), (1.10) we can rewrite integrals (1.8) as follows:

$$\begin{aligned}\delta A_1 = \iint_{(\Omega)} [(p_{1x} \delta u_1 + p_{1y} \delta v_1 + p_{1z} \delta w_1) D_1(x, y) + \\ + (p_{2x} \delta u_2 + p_{2y} \delta v_2 + p_{2z} \delta w_2) D_2(x, y)] \, dx\,dy \qquad (1.11)\end{aligned}$$

Making use of formulas (1.1), we express now the displacement variations of the points on the plate faces in terms of variations of basic variables $u_0, v_0, \ldots, w_0'$ and their derivatives. Making use also of the formulas which determine the values of $\chi_x^{(n)}, \ldots, \varphi_x^{(n)}$, we can transform the elementary work done by the face forces (1.11) as follows:

$$\delta A_1 = \sum_{n=0}^{\infty} \iint_{(\Omega)} \left\{ \frac{(-1)^n}{(2n)!} [(h_1^{2n} D_1 p_{1x} + h_2^{2n} D_2 p_{2x}) \delta\chi_x^{(n)} + (h_1^{2n} D_1 p_{1y} + h_2^{2n} D_2 p_{2y}) \delta\chi_y^{(n)} + \right.$$
$$+ (h_1^{2n} p_{1z} D_1 + h_2^{2n} D_2 p_{2z}) \delta\xi^{(n)}] + \frac{(-1)^n}{(2n+1)!} [(h_1^{2n+1} D_1 p_{1x} + h_2^{2n+1} D_2 p_{2x}) \delta\psi_x^{(n)} +$$
$$\left. + (h_1^{2n+1} D_1 p_{1y} + h_2^{2n+1} p_{2y} D_2) \delta\psi_y^{(n)} + (h_1^{2n+1} D_1 p_{1z} + h_2^{2n+1} D_2 p_{2z}) \delta\varphi^{(n)}] \right\} dx\, dy \quad (1.12)$$

Let us now calculate the elementary work of external forces applied to the cylindrical side surface. We shall denote the force applied to the unit area of the side surface as $\mathbf{q}_n$. The elementary work of these forces on the entire side surface is then expressed by the integral

$$\delta A_2 = \int_{h_2}^{h_1} dz \oint_{(L)} \mathbf{q}_n \cdot \delta\mathbf{u}\, ds = \int_{h_2}^{h_1} dz \oint_{(L)} (q_{nx}\delta u + q_{ny}\delta v + q_{nz}\delta w)\, ds \quad (1.13)$$

Introducing the static and hyperstatic notation of (1.1) into calculation of displacement variations, we have instead of (1.13) the following expression:

$$\delta A_2 = \sum_{n=0}^{\infty} \oint_{(L)} (R_x^{(n)} \delta\chi_x^{(n)} + R_y^{(n)} \delta\chi_y^{(n)} + Q^{(n)} \delta\xi^{(n)} - M_x^{(n)} \delta\psi_y^{(n)} +$$
$$+ M_y^{(n)} \delta\psi_x^{(n)} + W^{(n)} \delta\varphi^{(n)})\, ds \quad (1.14)$$

Here $R_x^{(0)}, \ldots, M_y^{(0)}$ are the static and $R_x^{(n)}, \ldots, W^{(n)}$ are the hyperstatic characteristics of the side load distribution through the plate thickness, expressed by formulas

$$R_x^{(n)} = \frac{(-1)^n}{(2n)!} \int_{h_2}^{h_1} q_{nx} z^{2n}\, dz, \qquad W^{(n)} = \frac{(-1)^n}{(2n+1)!} \int_{h_2}^{h_1} q_{nz} z^{2n+1} dz$$
$$-M_x^{(n)} = \frac{(-1)^n}{(2n+1)!} \int_{h_2}^{h_1} q_{ny} z^{2n+1} dz, \qquad Q^{(n)} = \frac{(-1)^n}{(2n)!} \int_{h_2}^{h_1} q_{nz} z^{2n} dz \quad (1.15)$$

The expressions for $R_y^{(n)}$ and $M_y^{(n)}$ are analogous.

The principle of minimum potential energy $\delta\Pi - \delta A_1 - \delta A_2 = 0$, after (1.7), (1.12) and (1.14) are allowed for, can be written as:

$$\sum_{n=0}^{\infty} \oint_{(L)} \{(n_x T_x^{(n)} + n_y S^{(n)} - R_x^{(n)}) \delta\chi_x^{(n)} + (n_x S^{(n)} + n_y T_y^{(n)} - R_y^{(n)}) \delta\chi_y^{(n)} +$$
$$+ (n_x N_x^{(n)} + n_y N_y^{(n)} - Q^{(n)}) \delta\xi^{(n)} + (n_x G_x^{(n)} + n_y H^{(n)} - M_y^{(n)}) \delta\psi_x^{(n)} +$$
$$+ (n_x H^{(n)} + n_y G_y^{(n)} + M_x^{(n)}) \delta\psi_y^{(n)} + (n_x \Gamma_x^{(n)} + n_y \Gamma_y^{(n)} - W^{(n)}) \delta\varphi^{(n)}\}\, ds -$$
$$- \sum_{n=0}^{\infty} \iint_{(\Omega)} \left\{ \left[\partial_1 T_x^{(n)} + \partial_2 S^{(n)} + \Gamma_x^{(n-1)} + \frac{(-1)^n}{(2n)!} (h_1^{2n} p_{1x} D_1 + h_2^{2n} p_{2x} D_2) \right] \delta\chi_x^{(n)} + \right.$$
$$+ \left[\partial_1 S^{(n)} + \partial_2 T_y^{(n-1)} + \Gamma_y^{(n)} + \frac{(-1)^n}{(2n)!} (h_1^{2n} p_{1y} D_1 + h_2^{2n} p_{2y} D_2) \right] \delta\chi_y^{(n)} +$$
$$+ \left[\partial_1 N_x^{(n)} + \partial_2 N_y^{(n)} - Z_f^{(n-1)} + \frac{(-1)^n}{(2n)!} (h_1^{2n} p_{1z} D_1 + h_2^{2n} p_{2z} D_2) \right] \delta\xi^{(n)} + \quad (1.16)$$

$$+\left[\partial_1 G_x^{(n)} + \partial_2 H^{(n)} - N_x^{(n)} + \frac{(-1)^n}{(2n+1)!}\,(h_1^{2n+1} p_{1x} D_1 + h_2^{2n+1} p_{2x} D_2)\right] \delta\psi_x^{(n)} + \qquad \text{(cont.)}$$

$$+\left[\partial_1 H^{(n)} + \partial_2 G_y^{(n)} - N_y^{(n)} + \frac{(-1)^n}{(2n+1)!}\,(h_1^{2n+1} p_{1y} D_1 + h_2^{2n+1} p_{2y} D_2)\right] \delta\psi_y^{(n)} +$$

$$+\left[\partial_1 \Gamma_x^{(n)} + \partial_2 \Gamma_y^{(n)} - Z_l^{(n)} + \frac{(-1)^n}{(2n+1)!}\,(h_1^{2n+1} p_{1z} D_1 + h_2^{2n+1} p_{2z} D_2)\right] \delta\varphi^{(n)}\Big\}\, dx\, dy = 0$$

$$(\Gamma_x^{(-1)} = \Gamma_y^{(-1)} = Z_l^{(-1)} \equiv 0)$$

The coefficients of the variations $\delta\chi_x^{(n)}, \ldots, \delta\varphi^{(n)}$ in the double integral (1.16) become zeros because of the equilibrium equations in terms of the stresses. We shall show it in the case of bracket expression next to variation $\delta\chi_x^{(n)}$; using (1.14) from [3] we have

$$\partial_1 T_x^{(n)} + \partial_2 S^{(n)} + \Gamma_x^{(n-1)} + \frac{(-1)^n}{(2n)!}\,(h_1^{2n} p_{1x} D_1 + h_2^{2n} p_{2x} D_2) = \frac{(-1)^n}{(2n)!}\left(\frac{\partial}{\partial x}\int_{h_2}^{h_1} \sigma_x z^{2n}\, dz + \right.$$

$$\left. + \frac{\partial}{\partial y}\int_{h_2}^{h_1} \tau_{xy} z^{2n}\, dz\right) + \frac{(-1)^n}{(2n-1)!}\int_{h_2}^{h_1} \tau_{zx} z^{2n-1} dz + \frac{(-1)^n}{(2n)!}\,(h_1^{2n} p_{1x} D_1 + h_2^{2n} p_{2x} D_2) \qquad (1.17)$$

On the other hand, as h_1 and h_2 are functions of variables x and y,

$$\frac{\partial}{\partial x}\left(\int_{h_2}^{h_1} \sigma_x z^{2n} dz\right) = \int_{h_2}^{h_1} \frac{\partial \sigma_x}{\partial x} z^{2n} dz + (\sigma_x)_{z=h_1} h_1^{2n} \frac{\partial h_1}{\partial x} - (\sigma_x)_{z=h_2} h_2^{2n} \frac{\partial h_2}{\partial x}$$

$$\frac{\partial}{\partial y}\left(\int_{h_2}^{h_1} \tau_{xy} z^{2n} dz\right) = \int_{h_2}^{h_1} \frac{\partial \tau_{xy}}{\partial y} z^{2n} dz + (\tau_{xy})_{z=h_1} h_1^{2n} \frac{\partial h_1}{\partial y} - (\tau_{xy})_{z=h_2} h_2^{2n} \frac{\partial h_2}{\partial y}$$

$$\int_{h_2}^{h_1} \frac{d\tau_{zx}}{dz} z^{2n} dz = (\tau_{zx})_{z=h_1} h_1^{2n} - (\tau_{zx})_{z=h_2} h_2^{2n} - 2n \int_{h_2}^{h_1} \tau_{zx} z^{2n-1} dz \qquad (1.18)$$

Substituting these relations into the considered bracket expression (1.17) we have

$$\partial_1 T_x^{(n)} + \partial_2 S^{(n)} + \Gamma_x^{(n-1)} + \frac{(-1)^n}{(2n)!}\,(h_1^{2n} p_{1x} D_1 + h_2^{2n} p_{2x} D_2) = \qquad (1.19)$$

$$= \frac{(-1)^n}{(2n)!}\left\{\int_{h_2}^{h_1}\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) z^{2n} dz + \left[(\sigma_x)_{z=h_1} \frac{\partial h_1}{\partial x} + (\tau_{xy})_{z=h_1} \frac{\partial h_1}{\partial y} - \right.\right.$$

$$\left.\left. - (\tau_{zx})_{z=h_1} + p_{1x} D_1\right] h_1^{2n} - \left[(\sigma_x)_{z=h_2} \frac{\partial h_2}{\partial x} + (\tau_{xy})_{z=h_2} \frac{\partial h_2}{\partial y} - (\tau_{zx})_{z=h_2} + p_{2x} D_2\right] h_2^{2n}\right\} = 0$$

since the left-hand part of the differential equation of stress equilibrium (volume forces are assumed nil) is under the integral sign, and the square brackets next to h_1^{2n} and h_2^{2n} become zeros because of the conditions on plate faces (cf [1]). Thus, the principle of minimum potential energy, expressed by integrals (1.16), leads to the following relation:

$$\oint_{(L)} \sum_{n=0}^{\infty} [(n_x T_x^{(n)} + n_y S^{(n)} - R_x^{(n)})\,\delta\chi_x^{(n)} + (n_x S^{(n)} + n_y T_y^{(n)} - R_y^{(n)})\,\delta\chi_y^{(n)} +$$

$$+ (n_x N_x^{(n)} + n_y N_y^{(n)} - Q^{(n)})\,\delta\xi^{(n)} + (n_x G_x^{(n)} + n_y H^{(n)} - M_y^{(n)})\,\delta\psi_x^{(n)} +$$

$$+ (n_x H^{(n)} + n_y G_y^{(n)} + M_x^{(n)})\,\delta\psi_y^{(n)} + (n_x \Gamma_x^{(n)} + n_y \Gamma_y^{(n)} - W^{(n)})\,\delta\varphi^{(n)}]\, ds = 0 \qquad (1.20)$$

which yields both the geometric conditions

$$\delta\chi_x^{(n)} = 0, \quad \delta\chi_y^{(n)} = 0, \quad \delta\xi^{(n)} = 0, \quad \delta\psi_x^{(n)} = 0, \quad \delta\psi_y^{(n)} = 0, \quad \delta\varphi^{(n)} = 0 \qquad (1.21)$$

and the force boundary conditions of the plate contour

$$\begin{gathered} n_x T_x^{(n)} + n_y S^{(n)} = R_x^{(n)}, \quad n_x G_x^{(n)} + n_y H^{(n)} = M_y^{(n)}, \quad n_x N_x^{(n)} + n_y N_y^{(n)} = Q^{(n)} \\ n_x S^{(n)} + n_y T_y^{(n)} = R_y^{(n)}, \quad n_x H^{(n)} + n_y G_y^{(n)} = -M_x^{(n)}, \quad n_x \Gamma_x^{(n)} + n_y \Gamma_y^{(n)} = W^{(n)} \\ (n = 0, 1, 2, \ldots) \end{gathered} \qquad (1.22)$$

Conditions (1.22) are derived from integral (1.20) by equating to zero the coefficients of independent variations $\delta\chi_x^{(n)}$, $\delta\chi_y^{(n)}$, $\delta\xi^{(n)}$, $\delta\psi_x^{(n)}$, $\delta\psi_y^{(n)}$, $\delta\varphi^{(n)}$ $(n = 0, 1, 2, \ldots)$. Conditions (1.21) and (1.22) were obtained earlier in [3] (formulas (1.17), (2.21), (1.18) and (2.12)) for a plate of uniform thickness. We have now shown that boundary conditions remain in the same form for a plate of variable thickness.

To obtain the differential equations of plate equilibrium we shall apply the equilibrium conditions on the plate faces [1], i.e. the Cauchy formulas. It must be remembered that the directional cosines of the normal to the plate faces $z = h_i(x, y)$ are

$$\begin{gathered} n_{1x} = -\frac{\partial_1 h_1}{D_1}, \quad n_{1y} = -\frac{\partial_2 h_1}{D_1}, \quad n_{1z} = \frac{1}{D_1} \\ n_{2x} = \frac{\partial_1 h_2}{D_2}, \quad n_{2y} = \frac{\partial_2 h_2}{D_2}, \quad n_{2z} = -\frac{1}{D_2} \end{gathered} \qquad (1.23)$$

Substituting (1.23) into Cauchy formulas, expressing stresses in terms of displacements, using symbolic notation and taking $z = h_1$, we obtain the first set of equilibrium equations for a plate of variable thickness

$$\begin{gathered} \left[2C(h_1)\left(\partial_1 u_0 + \frac{\vartheta_0}{m-2}\right) - \frac{mh_1 S(h_1)}{m-2}\partial_1^2\vartheta_0 + S(h_1)\left(2\partial_1 u_0' + \frac{\vartheta_0'}{m-1}\right) - \right. \\ \left. - \frac{m\Lambda(h_1)}{2(m-1)}\partial_1^2\vartheta_0'\right]\partial_1 h_1 + \left[C(h_1)(\partial_1 v_0 + \partial_2 u_0) - \frac{mh_1 S(h_1)}{m-2}\partial_1\partial_2\vartheta_0 + S(h_1)(\partial_1 v_0' + \right. \\ \left. + \partial_2 u_0') - \frac{m\Lambda(h_1)}{2(m-1)}\partial_1\partial_2\vartheta_0'\right]\partial_2 h_1 - \left[S(h_1)(\partial_1 w_0' - \Delta u_0) - \frac{mh_1 C(h_1)}{m-2}\partial_1\vartheta_0 + \right. \\ \left. + C(h_1)(u_0' + \partial_1 w_0) - \frac{mh_1 S(h_1)}{2(m-1)}\partial_1\vartheta_0'\right] + \frac{D_1 p_{1x}}{\mu} = 0 \\ \left[S(h_1)(\partial_1 w_0' - \Delta u_0) - \frac{mh_1 C(h_1)}{m-2}\partial_1\vartheta_0 + C(h_1)(u_0' + \partial_1 w_0) - \right. \\ \left. - \frac{mh_1 S(h_1)}{2(m-1)}\partial_1\vartheta_0'\right]\partial_1 h_1 + \left[S(h_1)(\partial_2 w_0' - \Delta v_0) - \frac{mh_1 C(h_1)}{m-2}\partial_2\vartheta_0 + C(h_1)(v_0' + \partial_2 w_0) - \right. \\ \left. - \frac{mh_1 S(h_1)}{2(m-1)}\partial_2\vartheta_0'\right]\partial_2 h_1 - \left[2C(h_1)\left(w_0' + \frac{\vartheta_0}{m-2}\right) + \frac{mh_1 S(h_1)}{m-2}\Delta\vartheta_0 - S(h_1)\left\{2\Delta w_0 + \right.\right. \\ \left.\left. + \frac{(m-2)\vartheta_0'}{2(m-1)}\right\} - \frac{mh_1 C(h_1)}{2(m-1)}\vartheta_0'\right] + \frac{D_1 p_{1z}}{\mu} = 0 \end{gathered} \qquad (1.24)$$

The first and third equations only are written out in (1.24); the second equation can be obtained from the first when ∂_2 is substituted for ∂_1 (and ∂_1 for ∂_2), v_0 for u_0, v_0' for u_0' and p_{2x} for p_{1x}. The fourth, fifth and sixth equations are obtained from the first three equations by substitution of h_2, D_2 $-p_{2x}$, $-p_{2y}$, $-p_{2z}$ for h_1, D_1, p_{1x}, p_{1y}, p_{1z}, respectively. The operators $C(h_i)$, $S(h_i)$, $\Lambda(h_i)$ in (1.24) are as follows:

$$C(h_i) = \sum_{n=0}^{\infty} \frac{(-1)^n h_i^{2n}\Delta^n}{(2n)!}, \qquad S(h_i) = \sum_{n=0}^{\infty} \frac{(-1)^n h_i^{2n+1}\Delta^n}{(2n+1)!} \qquad (1.25)$$

$$\Lambda(h_i) = \sum_{n=0}^{\infty} \frac{(-1)^n h_i^{2n+3} \Delta^n}{(2n+1)!\,(2n+3)} \qquad (i = 1, 2) \qquad \text{(cont.)}$$

in other words, they are derived from operators c, s, λ (1.2) by substituting the parameter h_1 or h_2 for coordinate z.

Let us now consider the particular case of a plate symmetrical with respect to the original plane, i.e. the case when $h_1 = -h_2 = h$ and $D_1 = D_2 = D$. Taking linear combinations of the first and fourth equations, second and fifth, and third and sixth of (1.24), adding up and calculating these pairs of equations we obtain in this case the individual equations of the plate extension and compression problem (in variables u_0, v_0, w_0') and of the plate bending problem (in variables u_0, v_0, w_0). We shall write out, as an example, the equations for the bending problem

$$\left[S\left(2\partial_1 u_0' + \frac{\vartheta_0'}{m-1}\right) - \frac{m\Lambda\partial_1^2}{2(m-1)}\vartheta_0'\right]\partial_1 h + \left[S(\partial_1 v_0' + \partial_2 u_0') - \right.$$
$$\left. - \frac{m\Lambda\partial_1\partial_2}{2(m-1)}\vartheta_0'\right]\partial_2 h - C(u_0' + \partial_1 w_0) + \frac{mhS\partial_1}{2(m-1)}\vartheta_0' + \frac{D}{2\mu}(p_{1x} - p_{2x}) = 0 \qquad (1.26)$$

$$\left[C(u_0' + \partial_1 w_0) - \frac{mhS\partial_1}{2(m-1)}\vartheta_0'\right]\partial_1 h + \left[C(v_0' + \partial_2 w_0) - \frac{mhS\partial_2}{2(m-1)}\vartheta_0'\right]\partial_2 h -$$
$$- S\left[2\Delta w_0 + \frac{(m-2)\vartheta_0'}{2(m-1)}\right] + \frac{mhC}{2(m-1)}\vartheta_0' + \frac{D}{2\mu}(p_{1z} + p_{2z}) = 0$$

Only the first and third equations are written out in (1.26) for the bending of a thick plate of variable thickness; the second equation can be obtained from the first by suitably changing the letters and indices.

When the plate thickness is uniform, $\partial_1 h = \partial_2 h = 0$, $D = 1$ and equations (1.26) are simplified, they are given in [2–4]. Another particular case is considered in the Sect. which follows.

2. Plate with plane lower base. Let the lower surface of the plate lie in the plane $z = 0$, while $h_2 = 0$, $h_1 = h$, $D_2 = 1$ and $D_1 = D$. The fourth, fifth and sixth equations of plate equilibrium (equations as in (1.24)) can be now written as follows:

$$u_0' + \partial_1 w_0 + \frac{p_{0x}}{\mu} = 0, \qquad v_0' + \partial_2 w_0 + \frac{p_{0y}}{\mu} = 0$$
$$2w_0' + \frac{2\vartheta_0}{m-2} + \frac{p_{0z}}{\mu} = 0 \qquad (2.1)$$

By means of (2.1) we can eliminate the variables u_0', v_0', w_0' from the first three equations of (1.24) and derive three new equations for the displacements of the original plane

$$[m(2C - hS\partial_1^2)\partial_1 u_0 + (2C - mhS\partial_1^2)\partial_2 v_0 - \{m(S + hC)\partial_1^2 + 2S\partial_2^2\}w_0]\partial_1 h + [\{(m-1)C -$$
$$- mhS\partial_1^2\}\partial_2 u_0 + \{(m-1)C - mhS\partial_2^2\}\partial_1 v_0 - \{(m-2)S + mhC\}\partial_1\partial_2 w_0]\partial_2 h +$$
$$+ [(S + mhC)\partial_1(\partial_1 u_0 + \partial_2 v_0) - (m-1)S\Delta u_0 - mhS\Delta\partial_1 w_0] + {}^1/_2 K_x / \mu = 0$$

$$[(S + mhC)\partial_1(\partial_1 u_0 + \partial_2 v_0) - (m-1)S\Delta u_0 - mhS\Delta\partial_1 w_0]\partial_1 h + [(S + mhC)\partial_2(\partial_1 u_0 +$$
$$+ \partial_2 v_0) - (m-1)S\Delta v_0 - mhS\Delta\partial_2 w_0]\partial_2 h -$$
$$- m[hS\Delta(\partial_1 u_0 + \partial_2 v_0] - \Lambda\Delta^2 w_0] + {}^1/_2 K_z/\mu = 0 \qquad (2.2)$$

Only the first and third equations are written out in (2.2); the second equation is obtained from the first by a suitable change of letters and indices. The components K_x and K_z which depend on the external loading are determined by the following expressions:

$$K_x = [\{m\Lambda\partial_1^2 - 2(2m-1)S\}\partial_1 p_{0x} + (m\Lambda\partial_1^2 - 2S)\partial_2 p_{0y} + \{mhS\partial_1^2 - 2(m-2)C\} p_{0z}]\partial_1 h + [\{m\Lambda\partial_1^2 - 2(m-1)S\}\partial_2 p_{0x} + \{m\Lambda\partial_2^2 - 2(m-1)S\} \partial_1 p_{0y} + mhS\partial_1\partial_2 p_{0z}]\partial_2 h + [\{2(m-1)C - mhS\partial_1^2\} p_{0x} - mhS\partial_1\partial_2 p_{0y} + (m\Lambda\Delta - 2S)\partial_1 p_{0z}] + 2(m-1)Dp_{1x}$$

$$K_z = [\{2(m-1)C - mhS\partial_1^2\} p_{0x} - mhS\partial_1\partial_2 p_{0y} + (m\Lambda\Delta - 2S)\partial_1 p_{0z}]\partial_1 h + [-mhS\partial_1\partial_2 p_{0x} + \{2(m-1)C - mhS\partial_2^2\} p_{0y} + (m\Lambda\Delta - 2S)\partial_2 p_{0z}]\partial_2 h - [\{(m-2)S + mhC\}(\partial_1 p_{0x} + \partial_2 p_{0y}) - \{2(m-1)C + mhS\Delta\} p_{0z}] - 2(m-1)Dp_{1z} \qquad (2.3)$$

3. Problem of equilibrium of an axially symmetrical circular plate of variable thickness. If our thick plate is circular in the plan projection, it is more convenient to use cylindrical coordinates r, φ, z. The corresponding displacements will be written as u_r, v_φ and w. When the deformation is axially symmetrical, $v_\varphi = 0$ and the solution must be independent of the polar angle φ.

Let us consider as an example the case of a plate with a plane circular base the displacement of which will be written as u_{r0} and w_0. These functions will be assumed to be dependent only on the radius r, then

$$u_0 = u_{r0}\cos\varphi, \qquad v_0 = u_{r0}\sin\varphi \qquad (3.1)$$

The load components p_{0r}, p_{0z}, p_{1r}, p_{1z} as well, depend only on r, so that

$$p_{ix} = p_{ir}\cos\varphi, \qquad p_{iy} = p_{ir}\sin\varphi \qquad (i = 0, 2) \qquad (3.2)$$

The relations between the derivative functions are given by

$$\partial_1 = \cos\varphi\,\partial_r - \frac{\sin\varphi}{r}\partial_\varphi, \qquad \partial_2 = \sin\varphi\,\partial_r + \frac{\cos\varphi}{r}\partial_\varphi \qquad (3.3)$$

Here $\partial_r = \partial/\partial r$ and $\partial_\varphi = \partial/\partial\varphi$. It follows, therefore, from (3.3) that when $h_1 = h = h(r)$

$$\partial_1 h = \cos\varphi\frac{dh}{dr}, \qquad \partial_2 h = \sin\varphi\frac{dh}{dr} \qquad (3.4)$$

It is evident that

$$\Delta[f(r)\cos k\varphi] = \cos k\varphi\Delta_k[f(r)], \quad \Delta[f(r)\sin k\varphi] = \sin k\varphi\Delta_k[f(r)] \qquad (3.5)$$

$$\Delta_k = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{k^2}{r^2} \qquad (3.6)$$

Making use of (3.2)–(3.6) we express (2.2) in polar coordinates. In the case of axial symmetry we have

$$\left\{[2(m+1)C - mhS\Delta]\left(\partial_r + \frac{1}{r}\right) + [2(m-1)C_2 - mhS_2\Delta_2]\left(\partial_r - \frac{1}{r}\right)\right\} u_{r0}\frac{dh}{dr} - \left\{[(m+2)S + mhC]\Delta + [(m-2)S_2 + mhC_2]\left(\partial_r^2 - \frac{\partial_r}{r}\right)\right\} w_0\frac{dh}{dr} - 2[(m-2)S_1 + mhC_1]\Delta_1 u_{r0} - 2mhS_1\Delta_1\partial_r w_0 + \frac{K_r}{\mu} = 0$$

$$\{[mhC_1 - (m-2)S_1]\Delta_1 u_{r0} - mhS_1\Delta_1\partial_r w_0\}\frac{dh}{dr} - m[hS\Delta\left(\partial_r + \frac{1}{r}\right)u_{r0} - \Lambda\Delta^2 w_0] + \frac{K_z}{2\mu} = 0 \qquad (3.7)$$

The operators S_k, C_k and Λ_k are introduced here which are obtained from operators S, C and Λ by the substitution of operators Δ_k (3.6) for the Laplacians. The argument h is left out as in (2.2) and (2.3). The load components K_r and K_z which appear in (3.7) assume the following form after transformations (3.2)–(3.6) are applied to expressions (2.3):

$$K_r = \left\{\left[\frac{m}{2}\Lambda_2\Delta_2 - 2(m-1)S_2\right]\left(\partial_r - \frac{1}{r}\right)p_{0r} + \left[\frac{mh}{2}S_2\left(\partial_r{}^2 - \frac{\partial_r}{r}\right) - \right.\right.$$
$$\left.\left. - 2(m-2)C_2\right]p_{0z}\right\}\frac{dh}{dr} + [2(m-1)C_1 - mhS_1\Delta_1]\,p_{0r} +$$
$$+ (m\Lambda_1\Delta_1 - 2S_1)\,\partial_r p_{0z} + 2(m-1)\,Dp_{1r}$$
$$K_z = \{[2(m-1)C_1 - mhS_1\Delta_1]\,p_{0r} + (m\Lambda_1\Delta_1 - 2S_1)\,\partial_r p_{0z}\}\frac{dh}{dr} -$$
$$- [(m-2)S + mhC]\left(\partial_r + \frac{1}{r}\right)p_{0r} + [2(m-1)C + mhS\Delta]\,p_{0z} - 2(m-1)\,Dp_{1z} \qquad (3.8)$$

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CONTACT PROBLEM FOR AN ELASTIC INFINITE CONE

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An exact solution is given herein for the mixed axisymmetric problem of elasticity theory for an infinite cone. It is assumed that the shear stresses are zero on its whole boundary surface $\theta = \theta_1$, and the homogeneous conditions for the normal stresses and normal displacements are separated by the circle $\theta = \theta_1$, $r = 1$ (r, θ, φ are spherical coordinates).

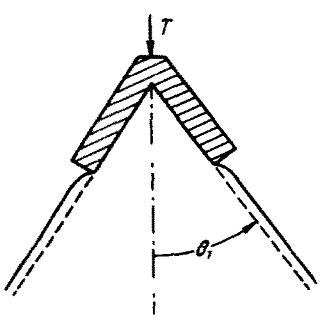

Fig. 1

Such problems arise, for example, in determining the state of stress of a cone compressed at its tip by a rigid cap of the same vertex angle as the cone (Fig. 1). They also arise in analyzing the intrusion of a conical die into a conical cavity made in an elastic space. The case $\theta_1 = {}^1/_2\,\pi$ corresponds to the symmetric indentation of a flat circular die into an elastic half-space.

It is assumed in formulating the problem that the elastic stress energy at the edge of the die and the